Maximum Cut of Graph by Semidefinite Relaxation

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Outline

1. What are Graphs Cuts and where we Apply them?
2. Mathematical Modelling of Max-Cut Problem
3. Relaxation by Semidefinite Programming
4. Semidfinite Solution

Figure: Example of Undirected Graph, $G = (V, E)$

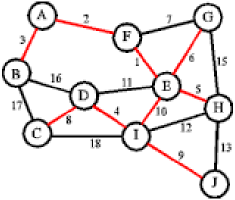


Figure: Example of Directed Graph, $G = (V, E)$

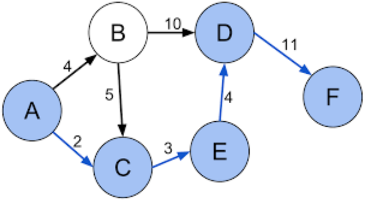
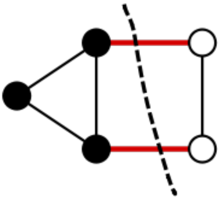


Figure: Example of Graph Cut



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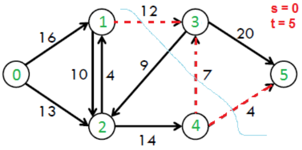

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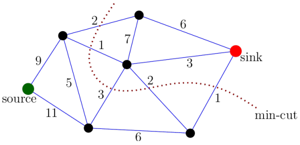

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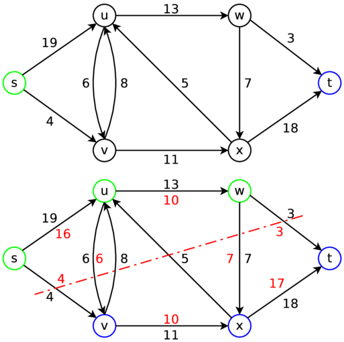

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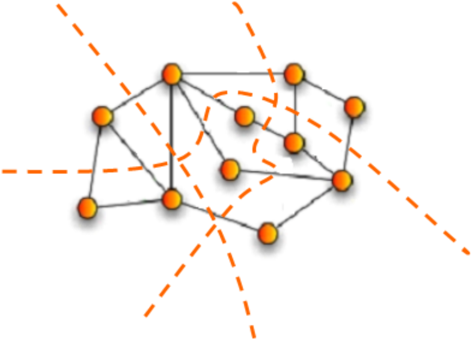

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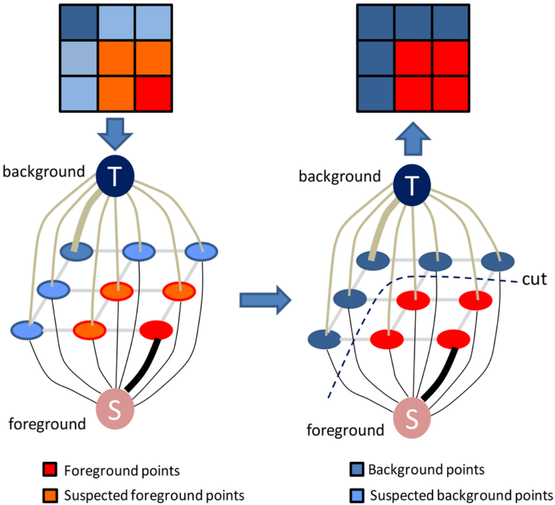

- Consider undirected $G(V, E)$ such that $|V| = n$ and $|E|$=m.
- Divide the vertex or node set V into S and $V \setminus S$ $(= \bar{S})$.
- For all edges $e = (i, j) \in E$ the weight $w_{ij} \in \mathbb{R}$ are known, $w_{ij} = w_{ji}$
- If $e = (i, j) \notin E$ then $w_{ij} = 0$, $w_{ii} = 0$, $\forall i \in V$.

- Associate a variable x_i for each node $i \in V$.
- Assign $x_i = 1$ if $i \in S$, and $x_i = -1$ if $i \in V \setminus S$, $i = 1, 2, \cdots, n$.
- Find an optimal S such that the cut value is maximum:

$$\frac{1}{4} \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij}(1 - x_i x_j)$$

$$Trace(AB) = \sum_i (AB)_{ii} = \sum_i \sum_j A_{ij} B_{ji} = \sum_j \sum_i B_{ji} A_{ij} = Trace(BA)$$

$$Trace\left((AB)^T\right) = \sum_i (B^T A^T)_{ii} = \sum_i \sum_j B^T_{ij} A^T_{ji} = \sum_i \sum_j A^T_{ji} B^T_{ij}$$

$$= \sum_i \sum_j A_{ij} B_{ji} = Trace(AB)$$

$$\max C \bullet X, \qquad s.t$$

$$A_i \bullet X = b_i \quad \forall i$$

$$X \succcurlyeq 0$$

$$C \bullet X = \langle C, X \rangle = Trace(C^T X) = Trace(CX)$$

$$LP: \quad \max \quad c^T x \quad s.t.$$
$$a_i^T x = b_i,\ i = 1, 2, \cdots, m.$$
$$x \in \mathbb{R}^n_+ = \{x \in \mathbb{R}^n \big| x_i \geq 0\}$$

$$A_i = \begin{pmatrix} a_{i1} & 0 & \cdots & 0 \\ 0 & a_{i2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{in} \end{pmatrix},\ i = 1, 2, \cdots, m; \quad C = \begin{pmatrix} c_1 & 0 & \cdots & 0 \\ 0 & c_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c_n \end{pmatrix}$$

$$X = \begin{pmatrix} x_1 & 0 & \cdots & 0 \\ 0 & x_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_n \end{pmatrix}$$

LPs are SDPs

$$LP: \quad \max \quad c^T x \quad s.t.$$
$$a_i^T x = b_i,\ i = 1, 2, \cdots, m.$$
$$x \in \mathbb{R}_+^n = \{x \in \mathbb{R}^n \big| x_i \geq 0\}$$
$$\Downarrow$$
$$SDP: \quad \max \quad C \bullet X \quad s.t.$$
$$A_i \bullet X = b_i,\ i = 1, 2, \cdots, m.$$
$$X \succcurlyeq 0$$

Define $y_i = (0, 0, \cdots, x_i)^T$ and $X_{ij} = y_i^T y_j$

$$L_{ij} = \begin{cases} \sum_k w_{ik} \ \text{if } i = j \\ -w_{ij} \text{ if } i \neq j \end{cases}$$

$$\begin{aligned} \frac{1}{4}\sum_{i=1}^n \sum_{j=1}^n w_{ij}(1 - X_{ij}) &= \frac{1}{4}\left(\sum_{i=1}^n \sum_{j=1}^n w_{ij} - \sum_{i=1}^n \sum_{j=1}^n w_{ij} X_{ij}\right) \\ &= \frac{1}{4}\left(\sum_i \left(\sum_j w_{ij}\right) + \sum_{i,i\neq j} \sum_{j,j\neq i} L_{ij} X_{ij}\right) \\ &= \frac{1}{4}\left(\sum_i L_{ii} X_{ii} + \sum_{i\neq j} L_{ij} X_{ij}\right) \\ &= \frac{1}{4}\langle L, X\rangle \\ &= \frac{1}{4} L \bullet X \end{aligned}$$

$$\max \frac{1}{4} L \bullet X, \qquad s.t$$
$$X \bullet e_i e_i^T = 1 \quad \forall i$$
$$X \succcurlyeq 0$$

$X_{ii} = X \bullet e_i e_i^T$ and e_i is the i-th coordinate vector.