# Linear stability of double diffusion convection

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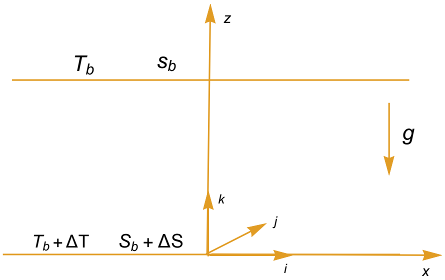
# Introduction

Double diffusion convection occurs when :

- Two components with different diffusion coefficients
- Opposing effects on the vertical density gradient



**Navier-Stokes equations**

$$\rho\Big[\frac{\partial V}{\partial t} + (V.\nabla)V\Big] = \nabla p + \mu\nabla^2 V + \rho F$$

Note: Neglect dissipation of heat due to viscosity

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho V) = 0$$

**Heat equation**

$$\frac{\partial T}{\partial t} + (V.\nabla)T = k_T \nabla^2 T$$

**Salinity equation**

$$\frac{\partial S}{\partial t} + (V.\nabla)S = k_S \nabla^2 S$$

# Perturbation Equations

$$\rho_1(T,S) = \rho_0(-\alpha_0 T_1 + \beta_0 S_1) \tag{1}$$

$$\rho_b \frac{\partial \bar{v}}{\partial t} = -\bar{\nabla} p_1 + \mu \nabla^2 \bar{v} + \rho_0 \left(\alpha_0 T_1 - \beta_0 S_1\right) g \bar{k} \tag{2}$$

$$\frac{\partial T_1}{\partial t} = \frac{\nabla T}{d} v_z + k_T \left( \frac{\partial^2 T_1}{\partial x^2} + \frac{\partial^2 T_1}{\partial z^2} \right) \tag{3}$$

$$\frac{\partial S_1}{\partial t} = \frac{\nabla S}{d} v_z + k_T \left( \frac{\partial^2 S_1}{\partial x^2} + \frac{\partial^2 S_1}{\partial z^2} \right) \tag{4}$$

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_z}{\partial z} = 0 \tag{5}$$

We take the curl of the N-S equation (2) to obtain

$$\rho_0 \frac{\partial \omega_y}{\partial t} = \mu \nabla^2 \omega_y + g \frac{\partial \rho_1}{\partial x} \tag{6}$$

where

$$\omega_y = \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \tag{7}$$

We introduce the stream function $\psi$, such that

$$v_x = \frac{\partial \psi}{\partial z}, \quad v_z = -\frac{\partial \psi}{\partial x} \tag{8a-b}$$

Equation (5) is then identically satisfied. Substituting (8a,b) and (1) into (6), (3) and (4) yield 3 equations in 3 unknowns given by

$$\left(\rho_0 \frac{\partial}{\partial t} - \mu \nabla^2 \right) \nabla^2 \psi = -\rho_0 g \alpha_0 \frac{\partial T_1}{\partial x} + \rho_0 \beta_0 g \frac{\partial S_1}{\partial x} \tag{9}$$

$$\left(\frac{\partial}{\partial t} - k_T \nabla^2\right) T_1 = -\frac{\Delta T}{d} \frac{\partial \psi}{\partial x} \tag{10}$$

$$\left(\frac{\partial}{\partial t} - k_T \nabla^2\right) S_1 = -\frac{\Delta S}{d} \frac{\partial \psi}{\partial x} \tag{11}$$

The dimensionless variables are introduce such that

$$t = \frac{\mathrm{d}^2}{k_T}\bar{t}, \quad v_x = \frac{k_T}{\mathrm{d}}\bar{v}_x, \quad v_z = \frac{k_T}{\mathrm{d}}\bar{v}_z, \quad \psi = k_T\bar{\psi},$$

$$T_1 = \Delta T\,\bar{T}_1, \quad S_1 = \Delta S\bar{S}_1, \quad x = \bar{x}\mathrm{d}, \quad z = \bar{z}\mathrm{d}. \tag{12}$$

In terms of dimensionless quantities, equations (9)-(11) becomes

$$\left(\frac{1}{\mathrm{Pr}}\frac{\partial}{\partial \bar{t}} - \bar{\nabla}^2\right)\bar{\nabla}^2\bar{\psi} = -\mathrm{Ra}\frac{\partial \bar{T}_1}{\partial \bar{x}} + \mathrm{Rs}\frac{\partial \bar{S}_1}{\partial \bar{x}} \tag{13}$$

$$\left(\frac{\partial}{\partial \bar{t}} - \bar{\nabla}^2\right)\bar{T}_1 = -\frac{\partial \bar{\psi}}{\partial \bar{x}} \tag{14}$$

$$\left(\frac{\partial}{\partial \bar{t}} - \tau\bar{\nabla}^2\right)\bar{S}_1 = -\frac{\partial \bar{\psi}}{\partial \bar{x}} \tag{15}$$

where $\tau = \frac{k_s}{k_T} < 1$. The Prandtl number Pr, thermal Raleigh number Ra and the salinity Raleigh number $R_s$ are defined as

$$\mathrm{Pr} = \frac{\nu}{k_T}, \quad \mathrm{Ra} = \frac{g\alpha_0 \Delta T d^3}{\nu k_T}, \quad R_s = \frac{g\beta_0 \Delta S d^3}{\nu k_T} \tag{16}$$

The dimensionless boundary conditions:
Zero normal velocity at the boundary $\bar{z} = 0$ and $\bar{z} = 1$:

$$\bar{v}_{\bar{z}}(\bar{x}, 0, \bar{t}) = 0 \Rightarrow \frac{\partial \bar{\psi}}{\partial \bar{x}}(\bar{x}, 0, \bar{t}) = 0 \Rightarrow \bar{\psi}(\bar{x}, 0, \bar{t}) = f(\bar{t}), (18)$$

$$\bar{v}_{\bar{z}}(\bar{x}, 1, \bar{t}) = 0 \Rightarrow \frac{\partial \bar{\psi}}{\partial \bar{x}}(\bar{x}, 1, \bar{t}) = 0 \Rightarrow \bar{\psi}(\bar{x}, 1, \bar{t}) = g(\bar{t}). (19)$$

We will also implement the condition of zero shear at the boundary $\bar{z} = 0$ and $\bar{z} = 1$ to get

$$\frac{\partial^2 \bar{\psi}}{\partial \bar{z}^2}(x, 0, t) = 0, \qquad \frac{\partial^2 \bar{\psi}}{\partial \bar{z}^2}(x, 1, t) = 0$$

# Perturbation analysis

The boundary conditions for Temperature and salt concentration are

$$\bar{T}_1(x, 0, t) = 0, \qquad \bar{S}_1(x, 0, t) = 0$$

$$\bar{T}_1(x, 1, t) = 0, \qquad \bar{S}_1(x, 1, t) = 0$$

We seek solutions for $\bar{\psi}$, $\bar{T}_1$ and $\bar{S}_1$ satisfying the boundary condition of the form:

$$\bar{\psi}(x, z, t) = A_n e^{\sigma t} \sin(\pi a x) \sin(\pi n z) \tag{20}$$

$$\bar{T}_1(x, z, t) = B_n e^{\sigma t} \cos(\pi a x) \sin(\pi n z) \tag{21}$$

$$\bar{S}_1(x, z, t) = C_n e^{\sigma t} \cos(\pi a x) \sin(\pi n z) \tag{22}$$

The dispersion relation

$$\sigma^3 + M\sigma^2 + N\sigma + Q = 0, \tag{23}$$

where

$$M = k^2(Pr\tau + 1) \tag{24}$$

$$N = (Pr + Pr\tau + \tau + 1)k^4 - \frac{1}{k^2}(a^2\pi^2 Pr(Ra - Rs)) \tag{25}$$

$$Q = Pr + \tau k^6 + a^2\pi^2 Pr(Rs - \tau Ra). \tag{26}$$

# Analysis of the dispersion relation roots

- Principle of exchange
  stabilities when $\sigma$ is real and the marginal states are characterized by $\sigma_r = 0$ and $\sigma_i = 0$

$$R_a{}^c = \frac{1}{\tau} R_s + \frac{27}{4}\pi^4$$

- Over stability
  The marginal state are characterized by $\sigma_r = 0$ and $\sigma_i \neq 0$

$$\omega^2 = (\tau + p_r\tau + p_r)k^4 - \frac{R_a - R_s}{k^2}\pi^2 a^2 p_r$$
$$= \frac{1}{k^2}[(\frac{1-\tau}{1+p_r})R_s\pi^2 a^2 p_s] - k^4\tau^2$$

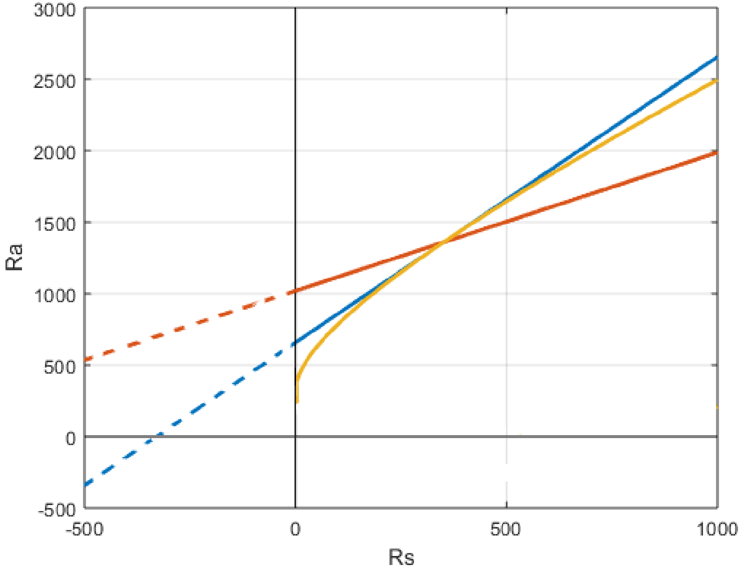
$$R_a{}^c = \frac{27\pi^4}{4p_r}(1+\tau)(\tau+p_r) + (\frac{\tau+p_r}{1+p_r})R_s$$

- The bifurcation point

$$FR_a^* = \frac{27}{4}\pi\frac{\tau^2}{p_r}(\frac{1+p_r}{1-\tau})$$

$$R_s^* = \frac{27}{4}\pi^4(\frac{p_r+\tau}{pr(1-\tau)})$$

# Stability Regions



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