OPTIMAL ASSIGNMENT OF BLOOD IN A BLOOD BANKING SYSTEM

J.P.F. Charpin*, A. Adewumi†

Study group participants

J.P.F. Charpin, M. Bashe, D. Hirpa , B. Okeke,
M. Seuneu Tchamga E.-L. Ujeneza

Industry representative
Aderemi Adewumi†

Abstract

Blood is required everyday in hospitals. This is a scarce resource and its management is complicated by the number of blood groups. A basic model is designed to manage a blood bank. The corresponding set of governing equations is derived and solution methods are investigated. Decision making is then expressed mathematically as a function of the blood bank stocks.

1 Introduction

Blood is required everyday in hospitals for a lot of different uses. Blood products are necessary during transfusions and may be classified as follows:

- Whole blood
- Red blood cells
- Blood plasma

*Mathematics Applications Consortium for Science and Industry, Department of Mathematics and Statistics, University of Limerick, Ireland. *e-mail: jean,charpin@ul.ie*

†School of Computer Science, University of KwaZulu-Natal, Durban, South Africa. *e-mail: adewumia@ukzn.ac.za*

- Platelets

Each of these products may be stored in adequate facilities for varying amounts of time, which extends from a few days to a few months/years [1, 2]. Blood banks, the departments of hospital where blood is managed, encounter numerous difficulties due to the complexity of the blood economy.

1. The demand for blood can vary considerably on a daily basis. In an emergency situation, vast amounts of blood may be requested within a very short amount of time. This comes on top of what would be the standard daily request.

2. The supply of blood may be erratic. Blood collection is uneven and depends on the generosity of donors.

3. Blood is a complex product. There are four main types of blood: O, A, B and AB. For convenience, blood type AB will be denoted C in the following. Blood types in the population vary from country to country. In general, O and A blood groups are dominant while B is slightly less important and C is much smaller [3]. In South Africa, blood type proportions are O, 46%, A, 37%, B, 14% and C, 4% [4].

   Table 1 shows the compatibility between the different red blood cell types. As is well known, O is the universal donor while AB is the universal receiver. The compatibilities are symmetrical for plasma types, see Table 2. For plasmas, O is the universal receiver while C is the universal donor.

Receiver/Donor	O	A	B	C
O	V	X	X	X
A	V	V	X	X
B	V	X	V	X
C	V	V	V	V

Table 1: Group compatibility for red blood cells

Receiver/Donor	O	A	B	C
O	V	V	V	V
A	X	V	X	V
B	X	X	V	V
C	X	X	X	V

Table 2: Group compatibility for plasma

4. Matters are further complicated by the Rhesus factor, another blood type parameter. It multiplies the number of blood group by two. In South Africa, 86% people have a positive Rhesus and 14% a negative Rhesus. The blood type repartition becomes $O+$, 39%, $O-$, 7%, $A+$, 32%, $A-$, 5%, $B+$, 12%, $B-$, 2%, $C+$, 3% and $C-$, 1% [4]. This modifies the compatibility between blood groups as shown in Table 3. As can be seen, $O-$ is the universal donor while $C+$ is the universal receiver.

Receiver/Donor	O-	O+	A-	A+	B-	B+	C-	C+
O-	V	X	X	X	X	X	X	X
O+	V	V	X	X	X	X	X	X
A-	V	V	V	V	X	X	X	X
A+	V	V	V	V	X	X	X	X
B-	V	V	X	X	V	V	X	X
B+	V	V	X	X	V	V	X	X
C-	V	X	V	X	V	X	V	X
C+	V	V	V	V	V	V	V	V

Table 3: Group compatibility for red blood cells with Rhesus

5. When emergencies happen, patients are given $O-$ blood type as this may be received by anyone.

In an ideal world, each person should be allocated his/her own blood type. Blood demand may however not meet the amount of blood donated. An imbalance can rapidly appear and blood of certain types may reach dangerously low levels. To compensate for this, when possible, alternative blood types may be used. Since blood products also only have a limited stocking time, to avoid wasting vital resources, a careful management is necessary as well. This problem of managing a blood bank was submitted to the Study Group. The group worked in 3 different directions. First mass balances were determined and analysed for each type of blood. Depending on the value of blood stocks in the morning, the security of blood supplies for the rest of the day may be evaluated. These mass balances will be detailed in Section 2. A time dependent model will then be presented which accounts for the variation of stocks over a long period of time. This will be presented in Section 3. Finally, a decision making process is defined in Section 4.

As already observed, the problem of blood management is complex and some assumptions were made to make the problem simpler.

1. In the following, only red blood cells will be considered. The model can easily include other blood products but reducing the study to a single aspect of the problem will considerably simplify the model.

2. Blood types were reduced to four groups: the Rhesus factor was not considered. Here again, this does not affect the model but will considerably simplify the equations and highlight basic properties of the solution. Once again, the model developed below can easily be generalised.

3. Blood supply and demand were assumed to be proportional to the representation of blood groups in the population. In this case, only the global blood request for a given day is required to carry out simulations.

4. Emergencies: emergency allocation of blood will not be considered to start with. If an emergency does arise, no optimisation may be performed, hospital personnel have no choice and patients will straight away receive $O-$ blood.

5. The validity date of the blood products will not be considered.

2 Mass balance

In this section, the amount of blood required by a hospital is denoted by Δ. An approximate value of this parameter should (realistically) be available to the management of the blood bank. As already observed, the amount of blood needed for each blood type is assumed to be proportional to the representation of blood types in the population n_O, n_A, n_B and n_C, and these values may be written:

$$\begin{aligned}
\Delta_O &= n_O \Delta \ , \\
\Delta_A &= n_A \Delta \ , \\
\Delta_B &= n_B \Delta \ , \\
\Delta_C &= n_C \Delta \ .
\end{aligned}$$

Using the red blood cell compatibility rules described in Table 1, people of blood type O may only receive blood of type O while people with other blood types may receive blood from other groups. In mathematical terms, the compatibility table may be expressed as follows:

$$\Delta_O = D_{OO} \ , \tag{1}$$

$$\Delta_A = D_{OA} + D_{AA} \ , \tag{2}$$

$$\Delta_B = D_{OB} + D_{BB} \ , \tag{3}$$

$$\Delta_C = D_{OC} + D_{AC} + D_{BC} + D_{CC} \ , \tag{4}$$

where D_{XY} represents the amount of blood of type X given to people of type Y. Equations (1-4) highlight where optimisation may take place: apart from type O, all blood groups may be substituted with at least another type of blood. Optimisation then consists in choosing the best replacement method. To start with, all blood quantities D_{XY} should be expressed as functions of V_O, V_A, V_B and V_C, the amount of blood of each type available in the blood bank. The simplest model is a linear relationship:

$$D_{XY} \propto V_X \ .$$

If there is a lot of type X blood in the bank, V_X is large and this blood type may be used to supply the request in blood of type Y. Conversely, if V_X is small, D_{XY} will be small and very little blood of type X will be used to substitute blood of type Y. Using this model, equations (1–4) may be rewritten:

$$\begin{aligned} \Delta_O &= \alpha_1 V_O \ , & (5)\\ \Delta_A &= \alpha_2 V_O + \beta_2 V_A \ , & (6)\\ \Delta_B &= \alpha_3 V_O + \gamma_3 V_B \ , & (7)\\ \Delta_C &= \alpha_4 V_O + \beta_4 V_A + \gamma_4 V_B + \delta_4 V_C \ , & (8) \end{aligned}$$

where α_i, β_i, γ_i and δ_i are positive constants yet to be determined. The value of parameter α_1 is imposed by equation (5):

$$\alpha_1 = \frac{\Delta_O}{V_O} \ .$$

If $\alpha_1 > 1$, the quantity of blood needed for that given day is larger than the quantity of blood of type O available in the bank. In this situation, blood of type O must be imported from outside the system, otherwise there will be a shortage at some stage in the day. If no substitution is allowed, the self replacement rates for β_2, γ_3 and δ_4 may be expressed as

$$\beta_2^s = \frac{\Delta_A}{V_A} \ , \qquad \gamma_3^s = \frac{\Delta_B}{V_B} \ , \qquad \delta_4^s = \frac{\Delta_C}{V_C} \ , \qquad (9)$$

where the superscript denotes the self replacement. Here again, if the parameters have a value larger than 1, this indicates potential shortages of blood types. If all four parameters α_1, β_2, γ_3 and δ_4 are larger than 1, the amount of blood requested for the day is larger than the amount of blood available in the bank and additional blood should be injected in the system. In ideal situations these parameters should be as small as possible: this would mean that the blood bank is well stocked in all types of bloods.

The quantities Δ_i and V_i are known at the beginning of every day. The system (5-8) gives four relationships between the nine constants α_i, β_i, γ_i and δ_i, which means that four of these parameters may be written as a function of the other five and the quantities Δ_i and V_i. These remaining parameters will be used in the optimisation process.

The system of equations (5-8) highlights the difficulties created by the number of blood types. If the Rhesus factor was considered, there would be eight equations and twenty three constants, meaning there would be fifteen degrees of freedom in the system.

The evolution of the stocks in the blood bank will now be studied as a function of time in Section 3 and methods to calculate the parameters will be detailed in Section 4.

3 Dynamical system

The evolution of the blood quantities in the bank will now be considered. The dynamical mass balance for this system may be described by equations (10-13):

$$\frac{d\,V_O}{dt} = Q_O - (D_{OO} + D_{OA} + D_{OB} + D_{OC}) \ , \tag{10}$$

$$\frac{d\,V_A}{dt} = Q_A - (D_{AA} + D_{AC}) \ , \tag{11}$$

$$\frac{d\,V_B}{dt} = Q_B - (D_{BB} + D_{BC}) \ , \tag{12}$$

$$\frac{d\,V_C}{dt} = Q_C - D_{CC} \ . \tag{13}$$

For each time unit, typically $t = 1$ should correspond to one day, blood enters and leaves the system. The variations of the blood volume stocked in the blood bank will correspond to the amount of blood donated for each blood group, denoted Q_O, Q_A, Q_B and Q_C, minus the amount of blood used during the day. With the notation defined in the previous section, the quantity of blood leaving the system may be expressed as a function of D_{XY}. Using the model above, the governing equations become:

$$\begin{aligned}
\frac{d\,V_O}{dt} &= Q_O - (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)\, V_O \ , \\
\frac{d\,V_A}{dt} &= Q_A - (\beta_2 + \beta_4)\, V_A \ , \\
\frac{d\,V_B}{dt} &= Q_B - (\gamma_3 + \gamma_4)\, V_B \ , \\
\frac{d\,V_C}{dt} &= Q_C - \delta_4 V_C \ .
\end{aligned}$$

These equations have the following solutions

$$V_O(t) = \left(V_O^0 - \frac{Q_O}{\alpha_1+\alpha_2+\alpha_3+\alpha_4}\right) e^{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)t} + \frac{Q_O}{\alpha_1+\alpha_2+\alpha_3+\alpha_4}\ , \quad (14)$$

$$V_A(t) = \left(V_A^0 - \frac{Q_A}{\beta_2+\beta_4}\right) e^{-(\beta_2+\beta_4)t} + \frac{Q_A}{\beta_2+\beta_4}\ , \quad (15)$$

$$V_B(t) = \left(V_B^0 - \frac{Q_B}{\gamma_3+\gamma_4}\right) e^{-(\gamma_3+\gamma_4)t} + \frac{Q_B}{\gamma_3+\gamma_4}\ , \quad (16)$$

$$V_C(t) = \left(V_C^0 - \frac{Q_C}{\delta_4}\right) e^{-\delta_4 t} + \frac{Q_C}{\delta_4}\ , \quad (17)$$

where V_X^0 represents the amount of blood present in the blood bank at the beginning of the day. If all the parameters V_X^0 and Q_X remained constant for long enough, the blood volumes in the blood bank would reach their asymptotic values:

$$V_O^l = \frac{Q_O}{\alpha_1+\alpha_2+\alpha_3+\alpha_4}\ , \quad V_A^l = \frac{Q_A}{\beta_2+\beta_4}\ , \quad V_B^l = \frac{Q_B}{\gamma_3+\gamma_4}\ , \quad V_C^l = \frac{Q_C}{\delta_4}\ ,$$

where the superscript l represents the long term value. The long term volumes of blood available in the blood bank should reflect the proportions of each blood type in the population (n_O, n_A, n_B, n_C). These proportions describe what should be the state of stocks in the blood bank in an ideal situation. This leads to another four equations relating the parameters α_i, β_i, γ_i and δ_i:

$$n_O = \frac{V_O^l}{V_O^l+V_A^l+V_B^l+V_C^l}\ ,$$

$$n_A = \frac{V_A^l}{V_O^l+V_A^l+V_B^l+V_C^l}\ ,$$

$$n_B = \frac{V_B^l}{V_O^l+V_A^l+V_B^l+V_C^l}\ ,$$

$$n_C = \frac{V_C^l}{V_O^l+V_A^l+V_B^l+V_C^l}\ .$$

Assuming the donated blood quantities Q_X are proportional to the blood proportion in the population, n_X, the equations above may be rewritten:

$$\frac{n_O - 1}{\alpha} + \frac{n_A}{\beta} + \frac{n_B}{\gamma} + \frac{n_C}{\delta} = 0 \,, \tag{18}$$
$$\frac{n_O}{\alpha} + \frac{n_A - 1}{\beta} + \frac{n_B}{\gamma} + \frac{n_C}{\delta} = 0 \,, \tag{19}$$
$$\frac{n_O}{\alpha} + \frac{n_A}{\beta} + \frac{n_B - 1}{\gamma} + \frac{n_C}{\delta} = 0 \,, \tag{20}$$
$$\frac{n_O}{\alpha} + \frac{n_A}{\beta} + \frac{n_B}{\gamma} + \frac{n_C - 1}{\delta} = 0 \,, \tag{21}$$

where

$$\alpha = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \;, \quad \beta = \beta_2 + \beta_4 \;, \quad \gamma = \gamma_3 + \gamma_4 \;, \quad \delta = \delta_4$$

and

$$n_O + n_A + n_B + n_C = 1 \;.$$

A straightforward analysis of the system (18-21) shows that there are only three independent parameters. Solving the system leads to the three equations:

$$\alpha = \beta = \gamma = \delta \;. \tag{22}$$

In the following, α, β, and γ will be expressed as a function of δ_4. Equations (5-8) and (22) may be written in a matrix form as:

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ V_O & 0 & 0 & V_A & 0 & 0 & 0 \\ 0 & V_O & 0 & 0 & 0 & V_B & 0 \\ 0 & 0 & V_O & 0 & V_A & 0 & V_B \end{pmatrix} \begin{pmatrix} \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \beta_2 \\ \beta_4 \\ \gamma_3 \\ \gamma_4 \end{pmatrix} = \begin{pmatrix} \delta_4 - \alpha_1 \\ \delta_4 \\ \delta_4 \\ \Delta_A \\ \Delta_B \\ \Delta_C - \delta_4 V_c \end{pmatrix} .$$

There are 6 equations and 7 unknowns. One of the unknowns will have to be used as a parameter. A priori, any of the unknown could be used but a straightforward analysis shows that for all choices, the resulting 6×6 matrix is singular, so a minimum of two unknowns should be used as parameters. The best candidates would be β_2 and γ_3 as they are the only unknowns for which some information is available as shown in (9). In total, the equations above provide 6 independent relationships between the α_i, β_i, γ_i and δ_4:

$$\begin{aligned}
\alpha_1 &= \frac{\Delta_O}{V_O} , \\
\alpha_2 &= \frac{\Delta_A - \beta_2 V_A}{V_O} , \\
\alpha_3 &= \frac{\Delta_B - \gamma_3 V_B}{V_O} , \\
\alpha_4 &= \delta_4 - \frac{\Delta_O + \Delta_A + \Delta_B}{V_O} + \frac{\beta_2 V_A + \gamma_3 V_B}{V_O} , \\
\beta_4 &= \delta_4 - \beta_2 , \\
\gamma_4 &= \delta_4 - \gamma_3 .
\end{aligned}$$

All parameters should be positive which implies the following inequalities:

$$\beta_2 \leq \frac{\Delta_A}{V_A} , \quad \gamma_3 \leq \frac{\Delta_B}{V_B} , \quad \delta_4 \leq \frac{\Delta_O + \Delta_A + \Delta_B}{V_O} - \frac{\beta_2 V_A + \gamma_3 V_B}{V_O} , \quad \beta_2 \leq \delta_4 , \quad \gamma_3 \leq \delta_4 .$$

The first two inequalities are directly related to the constraints on the blood bank and should be verified automatically: they imply that the coefficients β_2 and γ_3 are smaller than the self replacement rates calculated in Section 2. The last two inequalities show that the self replacement rates for blood groups A and B should be lower than the self replacement rate of group C.

These equations combined with (5-8) define six of the required parameters. The remaining parameters will be defined by optimising the decision making as will now be discussed.

4 Decision making

At the end of the first day, equations (14-17) provide the stocks of blood available in the bank

$$\begin{aligned}
V_O(1) &= \left(V_O^0 - \frac{Q_O}{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4} \right) e^{-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} + \frac{Q_O}{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4} , \\
V_A(1) &= \left(V_A^0 - \frac{Q_A}{\beta_2 + \beta_4} \right) e^{-(\beta_2+\beta_4)} + \frac{Q_A}{\beta_2 + \beta_4} , \\
V_B(1) &= \left(V_B^0 - \frac{Q_B}{\gamma_3 + \gamma_4} \right) e^{-(\gamma_3+\gamma_4)} + \frac{Q_B}{\gamma_3 + \gamma_4} , \\
V_C(1) &= \left(V_C^0 - \frac{Q_C}{\delta_4} \right) e^{-\delta_4} + \frac{Q_C}{\delta_4} .
\end{aligned}$$

Combined with equations (22), these volumes become

$$
\begin{aligned}
V_O(1) &= \left(V_O^0 - \frac{Q_O}{\delta_4}\right) e^{-\delta_4} + \frac{Q_O}{\delta_4} \, , \\
V_A(1) &= \left(V_A^0 - \frac{Q_A}{\delta_4}\right) e^{-\delta_4} + \frac{Q_A}{\delta_4} \, , \\
V_B(1) &= \left(V_B^0 - \frac{Q_B}{\delta_4}\right) e^{-\delta_4} + \frac{Q_B}{\delta_4} \, , \\
V_C(1) &= \left(V_C^0 - \frac{Q_C}{\delta_4}\right) e^{-\delta_4} + \frac{Q_C}{\delta_4}
\end{aligned}
$$

and the corresponding blood proportions in the bank may be expressed as

$$
\begin{aligned}
p_O &= \frac{V_O(1)}{V_O(1) + V_A(1) + V_B(1) + V_C(1)} \, , \\
p_A &= \frac{V_A(1)}{V_O(1) + V_A(1) + V_B(1) + V_C(1)} \, , \\
p_B &= \frac{V_B(1)}{V_O(1) + V_A(1) + V_B(1) + V_C(1)} \, , \\
p_C &= \frac{V_C(1)}{V_O(1) + V_A(1) + V_B(1) + V_C(1)} \, .
\end{aligned}
$$

Managing the blood bank efficiently requires getting these proportions at the end of the day as close as possible to the ideal proportions of the bank. This may be achieved by minimising the objective function

$$E = (p_O - n_O)^2 + (p_A - n_A)^2 + (p_B - n_B)^2 + (p_C - n_C)^2 \, .$$

This function only depends on the parameter δ_4. Values for the two parameters β_2 and γ_3 should also be provided by the optimisation process. Ideally, the values for these two quantities should be as close as possible to the self replacement values β_2^s and γ_4^s. The objective function above may then be modified to

$$E = (p_O - n_O)^2 + (p_A - n_A)^2 + (p_B - n_B)^2 + (p_C - n_C)^2 + \left(1 - \frac{\beta_2}{\beta_2^s}\right)^2 + \left(1 - \frac{\gamma_3}{\gamma_3^s}\right)^2 \, .$$

This function is only a possibility. Numerous other combinations could be considered to find optimal values for β_2, γ_3 and δ_4. As the expression for E is rather complex, numerical methods such as the gradient method or any other standard method should be considered to calculate the minimum. Once the minimum is calculated, values for the parameters β_2, γ_3 and δ_4 have been determined and all constants may be calculated. Decisions are then available for the blood bank manager. They may be expressed using the following proportions:

$$r_{OA} = \frac{D_{OA}}{\Delta_A}\,, \qquad r_{AA} = \frac{D_{AA}}{\Delta_A}\,,$$
$$r_{OB} = \frac{D_{OB}}{\Delta_B}\,, \qquad r_{BB} = \frac{D_{BB}}{\Delta_B}\,,$$
$$r_{OC} = \frac{D_{OC}}{\Delta_C}\,, \qquad r_{AC} = \frac{D_{AC}}{\Delta_C}\,, \qquad r_{BC} = \frac{D_{BC}}{\Delta_C}\,, \qquad r_{CC} = \frac{D_{CC}}{\Delta_C}\,,$$

where r_{XY} represents the percentage rate of blood of type X used to replace blood of type Y. This number should be given to the bank manager. For example, if $r_{OA} = 0.2$, 20% of the blood of type A required should be replaced by blood of type O, which corresponds to 1 in 5 allocations. These values can be calculated at the beginning of every day. If blood use and supply do not vary significantly for a significant period, these decisions will lead to the ideal proportion repartitions in the blood bank.

5 Conclusion and future work

The study group developed a model for the management of a blood bank. This model relates the proportion of each blood group in the population and the blood use and donations to the stocks available in the blood bank. Using a linear model, blood replacement coefficients may be defined. Mass balances and long term goals define six relationships between the nine coefficients. The last three may be estimated with an optimisation procedure minimising the difference between stocks and ideal situation in the short term and favouring replacement by blood of the same type. The model provides the proportion of type X blood that should be replaced with type Y blood and this information could be given to the manager.

In the future, the model could be further developed.

- The Rhesus factor should be included. This would not affect the complexity of the model but would significantly increase the number of equations. This would also make the optimisation process more difficult as there would be more parameters to optimise on.

- The present study is based on red blood cells only. This should be combined with other blood products.

- Emergencies should be included in the model. This would involve modifying the ideal proportions of blood in the bank when in the present situation, these proportions reflect exactly the repartition of blood groups in the population. Including emergencies would therefore make the model slightly more complex but the results presented here would not be significantly affected.

- The model should be modified to include indications about the validity date of the products. This aspect was not taken into account in the present study but could be included when choosing the form of the objective function.

- A linear model was chosen for simplicity here. The validity of the model should be investigated. Other types of relationships between the stocks and the amount of blood needed should be considered.

- The objective function will have to be studied carefully. Other forms could lead to more efficient management of existing resources.

- The model will have to be carefully tested using real data.

Acknowledgements

All contributors would like to thank Dr Aderemi Adewumi from the University of KwaZulu-Natal for introducing the problem and assisting in answering questions. J.P.F. Charpin acknowledges the support of the Mathematics Applications Consortium for Science and Industry (www.macsi.ul.ie) funded by the Science Foundation Ireland mathematics initiative grant 06/MI/005.

References

[1] Wikipedia. Blood bank. http://en.wikipedia.org/wiki/Blood.bank.

[2] Wikipedia. Blood product. http://en.wikipedia.org/wiki/Blood.product.

[3] LexiLine. Abo Blood types world distribution by people and country. http://www.lexiline.com/lexiline/lexi165.htm.

[4] South African National Blood Service. What's your type. http://www.sanbs.org.za/donors_whattype.htm.